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A novel formulation for the numerical computation of magnetization modes in complex micromagnetic systems

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ABSTRACT

The small oscillation modes in complex micromagnetic systems around an equilibrium are numerically evaluated in the frequency domain by using a novel formulation, which naturally preserves the main physical properties of the problem. The Landau–Lifshitz–Gilbert (LLG) equation, which describes magnetization dynamics, is linearized around a stable equilibrium configuration and the stability of micromagnetic equilibria is discussed. Special attention is paid to take into account the property of conservation of magnetization magnitude in the continuum as well as discrete model. The linear equation is recast in the frequency domain as a generalized eigenvalue problem for suitable self-adjoint operators connected to the micromagnetic effective field. This allows one to determine the normal oscillation modes and natural frequencies circumventing the difficulties arising in time-domain analysis. The generalized eigenvalue problem may be conveniently discretized by finite difference or finite element methods depending on the geometry of the magnetic system. The spectral properties of the eigenvalue problem are derived in the lossless limit. Perturbation analysis is developed in order to compute the changes in the natural frequencies and oscillation modes arising from the dissipative effects. It is shown that the discrete approximation of the eigenvalue problem obtained either by finite difference or finite element methods has a structure which preserves relevant properties of the continuum formulation. Finally, the generalized eigenvalue problem is solved for a rectangular magnetic thin-film by using the finite differences and for a linear chain of magnetic nanospheres by using the finite elements. The natural frequencies and the spatial distribution of the natural modes are numerically computed.

1. Introduction

The analysis and the determination of the resonances in micromagnetic systems are very important to understand the magnetization dynamics driven by radio-frequency external magnetic fields. Traditionally, the main experimental and theoretical studies concerning this problem are related to the classical ferromagnetic resonance experiments. In this situation, a small ferromagnetic particle is initially brought in the saturated state by means of a sufficiently strong external DC magnetic field which is applied along a given direction. Then, the particle is exposed to a radio-frequency (RF) external magnetic field. The amplitude of this RF field is usually much smaller than the DC component and the wavelength is in the range of centimeters, typically much larger than the characteristic dimension of the magnetic particle.

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The frequency of the applied RF field is slowly varied and the power absorbed by the magnetic particle is measured as a function of the applied field frequency. The peaks in this response curve reveal the excitation of certain magnetization normal oscillations connected to the resonance phenomena.

The response of the system exhibits a rich variety of behaviors depending on the strength of the applied RF field. For sufficiently small amplitudes, one expects the typical linear resonant response in which the excited modes are somehow decoupled. Conversely, when the RF power is increased above certain thresholds, the coupling of the oscillation modes due to nonlinear effects cannot be neglected.

In this respect, it was initially conjectured that a spatially uniform RF field would have certainly induced a spatially uniform magnetic response. Later, it became clear that, owing to the nonlinear nature of magnetization dynamics, at sufficiently high RF powers, spatially uniform motions could get coupled to certain spin-wave modes, giving rise to complicated nonuniform magnetization configurations [1,2]. Moreover, depending on the strength of the RF component, nonlinear dynamical phenomena such as foldover and spin-wave instabilities can also manifest [3,4].

For this reasons, the determination of the natural modes in generic micromagnetic systems is a fundamental step in the investigation of microwave-driven magnetization dynamics.

Another scenario where the normal oscillations around an equilibrium play a fundamental role is in the modelling of thermal fluctuations. In fact, thermal agitation tends to slightly perturb the equilibrium magnetization and therefore, from the analysis of the resonant response of the micromagnetic system, one can retrieve insightful information about fluctuation and dissipation processes.

Experimental observations of thermally-induced spectra have been carried out in the case of equilibrium magnetization configurations which are not necessarily spatially uniform, such as, for example, Landau configurations in ultra-thin magnetic films (see Refs. [5,6] and references therein).

The theoretical description of magnetization resonance phenomena has also been the focus of considerable research. Walker analyzed the case of saturated ellipsoidal particles in which the exchange interaction could be neglected [7]. On the other hand, Aharoni tackled the dual case of saturated magnetic nanospheres in which the exchange interaction was prevalent with respect to magnetostatics [8]. Brown completely analyzed the case of an infinite cylinder [9].

These fundamental works produced analytical techniques to determine the natural frequencies and modes, but their use was limited to the case of particles with very special shapes. Moreover, these approaches are applicable to the study of magnetization oscillations around saturated (spatially uniform) equilibrium configurations.

Recently, there has been growing interest in the computation of resonances for particles with generic shapes such as, for instance, rectangular thin-films or ferromagnetic prisms. To this end, numerical techniques based on micromagnetic simulations and Fourier analysis have been proposed [10–12]. In these works, magnetization dynamics around the equilibrium is excited with suitable field pulse and, from the Fourier analysis of the response, the normal modes are extracted. These methods have some disadvantages: (1) first, it is not trivial how to design a field pulse which excites all the modes of the system, then it is very difficult to distinguish the normal oscillations by analyzing the magnetization response; (2) in the case of low natural frequencies, the micromagnetic simulations should take very long time in order to have a good time resolution of the oscillation. For this problem, also methods based on the dynamical matrix have been proposed [13–16]. The latter approach is a numerical method based on the solution of a discrete eigenvalue problem whose unknowns are the spherical angles representing the deviations of the magnetization in each cell with respect to the ground state. This method has been applied to the study of rectangular geometries [13–15] and magnetic dipoles chains [16].

This paper proposes a general and effective approach to study and compute numerically the small oscillations around generic micromagnetic equilibria in magnetic systems with arbitrary geometries. Preliminary results relevant to some specific cases have been published in [17,18].

Small oscillations problems are usually formulated in terms of appropriate symmetric self-adjoint eigenvalue problems [19]. The matrices involved in such problems are related to the relevant terms of the energy in the system. In classical mechanics, small oscillation problems represent physical situations in which the total energy is conserved and there is continuous exchange between kinetic and potential energy.

On the other hand, small oscillations around micromagnetic equilibria have peculiar characteristics with respect to classical mechanical oscillations. The magnetization vector field has constant magnitude and only its direction can change in space and time. This occurs through precessional type oscillations. In order to take into account this property in the problem of small oscillations, special attention has to be paid in the linearization procedure. The issues of time-domain numerical simulations due to this peculiarity of micromagnetic oscillations and dynamics have been studied in details in Refs. [20–23]. Here we focus our attention on frequency domain analysis for the linear model. In this respect, we will reformulate the problem of small oscillations around micromagnetic equilibria as a generalized eigenvalue problem for self-adjoint operators connected to the micromagnetic effective field. These operators take into account the different kinds of energies of the micromagnetic system. This formulation can be used to compute magnetization oscillations around generic (spatially non-uniform) equilibrium configurations. In particular, we propose a general discretized model of the eigenvalue problem, based either on the finite difference or the finite element methods, which naturally preserves the structural properties of the continuum problem.

The structure of the paper is the following: in Section 2, we introduce and recall the properties of the free energy functional which describes the state of a generic magnetic body. Then, in Section 3, micromagnetic equilibria are analyzed. In Section 4, the problem of the conservative free oscillations of the micromagnetic system around an equilibrium configuration is formulated as suitable generalized eigenvalue problem and its spectral properties are studied. The stability and instability of micromagnetic equilibria are also addressed and, by using appropriate perturbation techniques, the small oscillations of the system are determined when dissipation effects are taken into account. In Section 5, the spatial discretization of the eigenvalue problem is addressed. Finally, in Section 6 the normal oscillations of two micromagnetic systems with different geometries are numerically computed in order to emphasize the effectiveness and the generality of the proposed approach.

2. Micromagnetic free energy

Let us assume that a region Ω is occupied by a magnetic body in contact with a thermal bath at constant temperature T. In the case of a system of magnetic bodies, the region Ω is the union of the regions occupied by each magnetic body. The magnetic state of the medium can be described by means of the magnetization vector field $\mathbf{M}(\mathbf{r},t)$ defined at each spatial location $\mathbf{r}\in\Omega$, for each time instant t and given temperature T.

Micromagnetic theory assumes the fundamental constraint $|\mathbf{M}(\mathbf{r},t)| = \mathrm{M_s}(T)$ for the magnetization, which is valid for temperatures T much smaller than the Curie temperature of the material [9,24], where $\mathrm{M_s}(T)$ is the saturation magnetization of the material at temperature T. For this reason, the state variable of the magnetic system becomes the magnetization unit-vector $\mathbf{m}(\mathbf{r},t) = \mathbf{M}/\mathrm{M_s}$ (the dependance on the temperature is understood) and the micromagnetic constraint is rewritten as

$$|\mathbf{m}| = 1. \tag{1}$$

In the sequel, to simplify the derivations, we will use dimensionless quantities.

The behaviour of the magnetic body is described by the normalized micromagnetic free energy [24] functional $g(\mathbf{m};\mathbf{h}_a) = G(\mathbf{M};\mathbf{H}_a)/(\mu_0 \mathrm{M_s^2} \mathrm{V}_\Omega)$, where $G(\mathbf{M};\mathbf{H}_a)$ is the Gibbs–Landau free energy functional in physical units, μ_0 is the vacuum magnetic permeability, V_Ω is the volume of the magnetic body and $\mathbf{h}_a = \mathbf{H}_a/\mathrm{M_s}$ is the normalized applied field. The functional g is formed by the sum of normalized exchange, magnetostatic, uniaxial anisotropy and Zeeman energies, respectively:

$$g(\mathbf{m};\mathbf{h}_a) = \frac{1}{\mathrm{V}_\Omega}\int_\Omega \left[\frac{\mathrm{l}_{\mathrm{ex}}^2}{2}(\nabla\mathbf{m})^2 - \frac{1}{2}\mathbf{h}_{\mathrm{m}}\cdot\mathbf{m} + \kappa_{\mathrm{an}}\left[1-(\mathbf{m}\cdot\mathbf{e}_{\mathrm{an}})^2\right] - \mathbf{h}_{\mathrm{a}}\cdot\mathbf{m}\right] dV, \tag{2}$$

where $\mathrm{l_{ex}} = \sqrt{2A/(\mu_0 M_s^2)}$ is the material exchange length, A is the exchange stiffness constant of the material, κ_{an} is the normalized uniaxial anisotropy constant, $\mathbf{e}_{\mathrm{an}}$ is the easy axis unit-vector and $\mathbf{h}_{\mathrm{m}}$ is the magnetostatic (demagnetizing) field, which is the solution of Maxwell's magnetostatic equations through the following boundary value problem:

$$\nabla\cdot\mathbf{h}_{\mathrm{m}} = -\nabla\cdot\mathbf{m}, \qquad \nabla\times\mathbf{h}_{\mathrm{m}} = 0, \tag{3}$$

$$\mathbf{n}\times[\mathbf{h}_{\mathrm{m}}]_{\partial\Omega} = \mathbf{0}, \qquad \mathbf{n}\cdot[\mathbf{h}_{\mathrm{m}}]_{\partial\Omega} = \mathbf{n}\cdot\mathbf{m}. \tag{4}$$

In Eqs. (3) and (4), we have denoted with $\mathbf{n}$ the outward normal to the boundary $\partial\Omega$ of the magnetic body, and with $[\mathbf{h}_m]_{\partial\Omega}$ the jump of the vector field $\mathbf{h}_m$ across $\partial\Omega$. It is well-known that the solution of the above linear boundary value problem (3) and (4) can be expressed as a function of the magnetization vector field in the following operator form:

$$\mathbf{h}_{\mathrm{m}}[\mathbf{m}] = -\mathcal{N}[\mathbf{m}] = \nabla\nabla\cdot\frac{1}{4\pi}\int_\Omega \frac{\mathbf{m}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dV_{\mathbf{r}'}. \tag{5}$$

3. Micromagnetic equilibria

The equilibrium configurations of the magnetic body, subject to the action of a static external magnetic field, can be found by appropriate minimization of the free energy functional (2) under the constraint expressed by Eq. (1). From the mathematical viewpoint, micromagnetic equilibria define magnetization vector fields $\mathbf{m}$ fulfilling the constraint (1) and such that the first-order variation of $g(\mathbf{m};\mathbf{h}_a)$ with respect to $\mathbf{m}$ is zero. The nature of these micromagnetic equilibria can be then studied by analyzing the convexity around the equilibrium of the free energy functional through its higher order derivatives.

Let us now consider a given magnetization configuration $\mathbf{m}_0(\mathbf{r})$. Then, let us perturb $\mathbf{m}_0$ such that $\mathbf{m}(\mathbf{r}) = \mathbf{m}_0(\mathbf{r}) + \delta\mathbf{m}(\mathbf{r})$.

In order to compute the derivatives from the expression of the free energy functional (2), one has to take into account that the variations $\delta\mathbf{m}$ of the magnetization cannot be arbitrary vector fields, since the total magnetization $\mathbf{m} = \mathbf{m}_0 + \delta\mathbf{m}$ must fulfill the constraint (1). In other terms, one can view Eq. (1) as defining an (infinite-dimensional) manifold $\mathcal{M}$. Therefore, the admissible variations $\delta\mathbf{m}$ are vector fields tangential to this manifold. For this reason, the derivatives of the free energy have to be computed directly along these vector fields. This special derivative is often referred to as Lie derivative [25].

Now we observe that any admissible variation $\delta\mathbf{m}(\mathbf{r})$ such that $|\mathbf{m}_0(\mathbf{r}) + \delta\mathbf{m}(\mathbf{r})| = 1$ can be obtained by performing a rotation of $\mathbf{m}_0(\mathbf{r})$ at each spatial location. The most general rotation can be generated in the following way: we consider an unconstrained vector field $\boldsymbol{\vartheta}(\mathbf{r})$. Then, we use the 3×3 matrix $\Lambda(\boldsymbol{\vartheta})$ such that

$$\Lambda(\boldsymbol{\vartheta})\mathbf{u} = \boldsymbol{\vartheta}\times\mathbf{u}, \quad \forall\mathbf{u}\in\mathbb{R}^3 \tag{6}$$

and compute the matrix exponential $\exp[\Lambda(\boldsymbol{\vartheta})]$. Indeed, the matrix product $\exp[\Lambda(\boldsymbol{\vartheta})]\cdot\mathbf{m}_0$ produces a rotation of $\mathbf{m}_0$ characterized by the angle $\theta = |\boldsymbol{\vartheta}_\perp|$ around the axis identified by the vector $\boldsymbol{\vartheta}_\perp$, where $\boldsymbol{\vartheta}_\perp$ is the component of $\boldsymbol{\vartheta}$ orthogonal to $\mathbf{m}_0$. Thus, the admissible variations are given by:

$$\mathbf{m}(\mathbf{r}) = \mathbf{m}_0(\mathbf{r}) + \delta\mathbf{m}(\mathbf{r}) = \exp[\Lambda(\varepsilon\boldsymbol{\vartheta}(\mathbf{r}))]\cdot\mathbf{m}_0(\mathbf{r}), \quad (7)$$

$$\delta\mathbf{m}(\mathbf{r}) = \{\exp[\Lambda(\varepsilon\boldsymbol{\vartheta}(\mathbf{r}))] - I\}\cdot\mathbf{m}_0(\mathbf{r}), \quad (8)$$

where ε is a parameter controlling the rotation amplitude and I is the identity matrix.

Now, in order to study the stationary points of the free energy functional, we are interested in its behaviour for small $\delta\mathbf{m}$ which can be studied by considering the following expansion:

$$g(\mathbf{m}_0 + \delta\mathbf{m}; \mathbf{h}_a) - g(\mathbf{m}_0; \mathbf{h}_a) = \varepsilon\,\delta g(\mathbf{m}_0, \delta\mathbf{m}) + \frac{\varepsilon^2}{2}\delta^2 g(\mathbf{m}_0, \delta\mathbf{m}) + O(\varepsilon^3), \quad (9)$$

where $\delta g(\mathbf{m}_0, \delta\mathbf{m})$ and $\delta^2 g(\mathbf{m}_0, \delta\mathbf{m})$ are respectively the first and the second-order variations of the free energy g, with respect to $\delta\mathbf{m}$ and evaluated on $\mathbf{m}_0$. We observe that $\delta g(\mathbf{m}_0, \delta\mathbf{m})$ is linear in $\delta\mathbf{m}$, whereas $\delta^2 g(\mathbf{m}_0, \delta\mathbf{m})$ is a quadratic form in $\delta\mathbf{m}$.

With the above definition of admissible variations, after appropriate manipulations, one can compute the expansion (9) with respect to a 'small' admissible variation connected with a small rotation amplitude $\varepsilon \ll 1$. In fact, by using the series expansion

$$\exp[\varepsilon\Lambda(\boldsymbol{\vartheta}(\mathbf{r}))] = I + \varepsilon\Lambda(\boldsymbol{\vartheta}) + \frac{\varepsilon^2}{2}\Lambda^2(\boldsymbol{\vartheta}) + O(\varepsilon^3) \quad (10)$$

the admissible variation (8) can be written as:

$$\delta\mathbf{m} = \varepsilon\Lambda(\boldsymbol{\vartheta})\cdot\mathbf{m}_0 + \frac{\varepsilon^2}{2}\Lambda^2(\boldsymbol{\vartheta})\cdot\mathbf{m}_0 + O(\varepsilon^3) \quad (11)$$

$$= \varepsilon\boldsymbol{\vartheta}\times\mathbf{m}_0 + \frac{\varepsilon^2}{2}\boldsymbol{\vartheta}\times(\boldsymbol{\vartheta}\times\mathbf{m}_0) + O(\varepsilon^3) \quad (12)$$

$$= \varepsilon\mathbf{v} + \frac{\varepsilon^2}{2}\boldsymbol{\vartheta}\times\mathbf{v} + O(\varepsilon^3), \quad (13)$$

where

$$\mathbf{v}(\mathbf{r}) = \boldsymbol{\vartheta}(\mathbf{r})\times\mathbf{m}_0(\mathbf{r}) \quad (14)$$

is a vector field pointwise perpendicular to the magnetization $\mathbf{m}_0(\mathbf{r})$.

By substituting the expansion (13) in the expression $g(\mathbf{m}_0 + \delta\mathbf{m}; \mathbf{h}_a) - g(\mathbf{m}_0; \mathbf{h}_a)$, one obtains after some algebra:

$$\begin{aligned} g(\exp[\varepsilon\Lambda(\boldsymbol{\vartheta})]\cdot\mathbf{m}_0; \mathbf{h}_a) - g(\mathbf{m}_0; \mathbf{h}_a) &= \varepsilon\,\delta g(\mathbf{m}_0, \mathbf{v}) + \frac{\varepsilon^2}{2}\delta^2 g(\mathbf{m}_0, \mathbf{v}) + O(\varepsilon^3) = \\ &= -\varepsilon\frac{1}{V_\Omega}\int_\Omega \mathbf{h}_{\text{eff}}(\mathbf{m}_0)\cdot\mathbf{v}\,dV + \varepsilon\frac{1}{V_\Omega}\int_{\partial\Omega}\frac{l_{\text{ex}}^2}{2}\frac{\partial\mathbf{m}_0}{\partial\mathbf{n}}\cdot\mathbf{v}\,dS + \\ &+ \frac{\varepsilon^2}{2}\frac{1}{V_\Omega}\int_\Omega\left\{\frac{l_{\text{ex}}^2}{2}|\nabla\mathbf{v}|^2 - \frac{1}{2}\mathbf{h}_m(\mathbf{v})\cdot\mathbf{v} - \kappa_{\text{an}}(\mathbf{v}\cdot\mathbf{e}_{\text{an}})^2 + \frac{h_0(\mathbf{r})}{2}|\mathbf{v}|^2\right\}dV + O(\varepsilon^3). \end{aligned} \quad (15)$$

Since the vector field $\boldsymbol{\vartheta}(\mathbf{r})$ is arbitrary, such is also $\mathbf{v}$ which, in turn, represents the generic admissible small variation of $\mathbf{m}_0$. In Eq. (15), the following notations have been also used:

$$\mathbf{h}_{\text{eff}}(\mathbf{m}_0) = l_{\text{ex}}^2\nabla^2\mathbf{m}_0 + \mathbf{h}_{\text{m}}(\mathbf{m}_0) + \kappa_{\text{an}}(\mathbf{e}_{\text{an}}\cdot\mathbf{m}_0)\mathbf{m}_0 + \mathbf{h}_{\text{a}}, \quad (16)$$

$$h_0(\mathbf{r}) = \mathbf{h}_{\text{eff}}(\mathbf{m}_0(\mathbf{r}))\cdot\mathbf{m}_0(\mathbf{r}). \quad (17)$$

By comparing Eq. (9) with Eqs. (15)–(17), it is easy to conclude that

$$\delta g(\mathbf{m}_0, \mathbf{v}) = -\frac{1}{V_\Omega}\int_\Omega \mathbf{h}_{\text{eff}}(\mathbf{m}_0)\cdot\mathbf{v}\,dV + \frac{1}{V_\Omega}\int_{\partial\Omega}\frac{l_{\text{ex}}^2}{2}\frac{\partial\mathbf{m}_0}{\partial\mathbf{n}}\cdot\mathbf{v}\,dS \quad (18)$$

$$\delta^2 g(\mathbf{m}_0, \mathbf{v}) = \frac{1}{V_\Omega}\int_\Omega\left\{\frac{l_{\text{ex}}^2}{2}|\nabla\mathbf{v}|^2 - \frac{1}{2}\mathbf{h}_m(\mathbf{v})\cdot\mathbf{v} - \kappa_{\text{an}}(\mathbf{v}\cdot\mathbf{e}_{\text{an}})^2 + \frac{h_0(\mathbf{r})}{2}|\mathbf{v}|^2\right\}dV. \quad (19)$$

Eq. (18) expresses the first-order variation of the free energy functional with respect to the vector field $\mathbf{v}$, evaluated on $\mathbf{m}_0$, whereas Eq. (19) expresses the second-order variation of g through the Hessian functional.

To summarize, magnetization vector fields $\mathbf{m}(\mathbf{r})$ on which $g(\mathbf{m}; \mathbf{h}_a)$ is evaluated, are elements of the (infinite dimensional) manifold $\mathcal{M}$ defined by the micromagnetic constraint (1). The vector fields $\mathbf{v}(\mathbf{r})$, which represent the small variations of $\mathbf{m}(\mathbf{r})$ around a given state $\mathbf{m}_0(\mathbf{r})$, belong to the tangent space of $\mathcal{M}$ at $\mathbf{m}_0$, which will be denoted by $T\mathcal{M}(\mathbf{m}_0)$, that is $\mathbf{v}(\mathbf{r})\cdot\mathbf{m}_0(\mathbf{r}) = 0$.

Eq. (16) defines the so called micromagnetic 'effective field' [9,24]. One can clearly see that, in general, the effective field is constituted by the sum of four terms associated with the corresponding contributions in the free energy: the exchange field $\mathbf{h}_{\text{ex}}$, the magnetostatic field $\mathbf{h}_{\text{m}}$, the anisotropy field $\mathbf{h}_{\text{an}}$ and the applied field $\mathbf{h}_{\text{a}}$:

$$\mathbf{h}_{\text{eff}}(\mathbf{m}, t) = \mathbf{h}_{\text{ex}} + \mathbf{h}_{\text{m}} + \mathbf{h}_{\text{an}} + \mathbf{h}_{\text{a}}(t). \quad (20)$$

In the general case, the explicit dependence of $\mathbf{h}_{\text{eff}}$ on time is related to the dependence on time of $\mathbf{h}_a$. The first three terms in Eq. (20) are linearly related to the vector field $\mathbf{m}$ through the following equations [24]:

$$\mathbf{h}_{\text{ex}}[\mathbf{m}] = l_{\text{ex}}^2 \nabla^2 \mathbf{m}, \tag{21}$$

$$\mathbf{h}_{\text{m}}[\mathbf{m}] = -\nabla\phi[\mathbf{m}] = -\nabla\nabla\cdot\frac{1}{4\pi}\int_\Omega \frac{\mathbf{m}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV_{\mathbf{r}'}, \tag{22}$$

$$\mathbf{h}_{\text{an}} = \kappa_{\text{an}}\mathbf{e}_{\text{an}}(\mathbf{r})(\mathbf{e}_{\text{an}}(\mathbf{r})\cdot\mathbf{m}(\mathbf{r})). \tag{23}$$

We now proceed to the derivation of the vector fields $\mathbf{m}_0$ representing equilibrium configurations. From Eq. (15), if one imposes that the first-order variation vanishes:

$$\begin{aligned}\delta g(\mathbf{m}_0,\mathbf{v}) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[g(\exp(\varepsilon\Lambda(\vartheta))\cdot\mathbf{m}_0;\mathbf{h}_a) - g(\mathbf{m}_0;\mathbf{h}_a)\right] \\ &= -\frac{1}{V_\Omega}\int_\Omega(\mathbf{m}_0\times\mathbf{h}_{\text{eff}}(\mathbf{m}_0))\cdot\vartheta\,dV + \frac{1}{V_\Omega}\oint_{\partial\Omega}\frac{l_{\text{ex}}^2}{2}\mathbf{m}_0\times\frac{\partial\mathbf{m}_0}{\partial\mathbf{n}}\cdot\vartheta\,dS = 0,\end{aligned} \tag{24}$$

by using the arbitrariness of $\vartheta(\mathbf{r})$, the Brown's equations for micromagnetic equilibria $\mathbf{m}_0(\mathbf{r})$ are derived[9]:

$$\mathbf{m}_0\times\mathbf{h}_{\text{eff}}(\mathbf{m}_0) = 0 \text{ in } \Omega, \quad \frac{\partial\mathbf{m}_0}{\partial\mathbf{n}} = 0 \text{ on } \partial\Omega. \tag{25}$$

The above equations characterize micromagnetic equilibria: the equilibrium configurations $\mathbf{m}_0$ are such that the magnetic effective torque is identically zero within the volume Ω and the homogeneous Neumann condition are fulfilled on the boundary $\partial\Omega$.

As mentioned before, the nature of the equilibrium $\mathbf{m}_0$ can be studied by means of the second-order variation of the free energy $g(\mathbf{m};\mathbf{h}_a)$. In fact, according to Eq. (24), the expansion (15) becomes:

$$\begin{aligned}g(\exp(\varepsilon\Lambda(\vartheta))\cdot\mathbf{m}_0;\mathbf{h}_a) - g(\mathbf{m}_0;\mathbf{h}_a) &= \frac{\varepsilon^2}{2}\delta^2 g(\mathbf{m}_0,\mathbf{v}) + O(\varepsilon^3) \\ &= \frac{\varepsilon^2}{2}\frac{1}{V_\Omega}\int_\Omega\left\{\frac{l_{\text{ex}}^2}{2}|\nabla\mathbf{v}|^2 - \frac{1}{2}\mathbf{h}_m(\mathbf{v})\cdot\mathbf{v} - \kappa_{\text{an}}(\mathbf{v}\cdot\mathbf{e}_{\text{an}})^2 + \frac{h_0(\mathbf{r})}{2}|\mathbf{v}|^2\right\}dV + O(\varepsilon^3).\end{aligned} \tag{26}$$

It can be readily seen that the equilibrium configuration minimizes the free energy if, for any admissible small variation (in the sense of Eq. (8)), the Hessian functional, evaluated on the equilibrium configuration $\mathbf{m}_0$, is a positive definite quadratic form in $\mathbf{v}$:

$$\delta^2 g(\mathbf{m}_0,\mathbf{v}) = \frac{1}{V_\Omega}\int_\Omega\left\{\frac{l_{\text{ex}}^2}{2}|\nabla\mathbf{v}|^2 - \frac{1}{2}\mathbf{h}_m(\mathbf{v})\cdot\mathbf{v} + \kappa_{\text{an}}(\mathbf{v}\cdot\mathbf{e}_{\text{an}})^2 + \frac{h_0(\mathbf{r})}{2}|\mathbf{v}|^2\right\}dV > 0 \quad \forall\mathbf{v}\neq 0\in T_{\mathcal{M}}(\mathbf{m}_0). \tag{27}$$

4. Small oscillations around an equilibrium configuration

We now proceed to the study of magnetization small oscillations around a micromagnetic equilibrium configuration $\mathbf{m}_0(\mathbf{r})$.

Magnetization dynamics is described by the Landau–Lifshitz–Gilbert equation [26] which, in normalized form, has the following expression:

$$\frac{\partial\mathbf{m}}{\partial t} = -\mathbf{m}\times\left(\mathbf{h}_{\text{eff}} - \alpha\frac{\partial\mathbf{m}}{\partial t}\right) \text{ in } \Omega. \tag{28}$$

In Eq. (28) $\mathbf{m}$ and $\mathbf{h}_{\text{eff}}$ are measured in units of the saturation magnetization M_s, α is the dimensionless Gilbert damping constant and time is measured in units of $(\gamma M_s)^{-1}$ (γ is the absolute value of the gyromagnetic ratio). Eq. (28) must be complemented with a boundary condition on $\partial\Omega$. In this respect, the magnetization $\mathbf{m}(\mathbf{r},t)$ is assumed to satisfy the following condition at the body surface [9]:

$$\frac{\partial\mathbf{m}}{\partial\mathbf{n}} = \mathbf{0}, \tag{29}$$

which is the condition expected when no surface anisotropy is present.

The effective field $\mathbf{h}_{\text{eff}}(\mathbf{m})$ can be expressed in terms of the Fréchet derivative of g. The Fréchet derivative of a generic functional $F(\mathbf{u})$ with respect to $\mathbf{u}$, denoted as $\delta F/\delta\mathbf{u}$, is defined such that [27]:

$$\delta F(\mathbf{u},\delta\mathbf{u}) = \lim_{\epsilon\to 0}\frac{F(\mathbf{u}+\epsilon\delta\mathbf{u}) - F(\mathbf{u})}{\epsilon} = \frac{1}{V_\Omega}\int_\Omega\frac{\delta F}{\delta\mathbf{u}}(\mathbf{u})\cdot\delta\mathbf{u}\,dV. \tag{30}$$

In this respect, since the homogeneous boundary condition (29) holds, from Eq. (18), it results [24]:

$$\frac{\delta g}{\delta \mathbf{m}}(\mathbf{m}) = -\mathbf{h}_{\text{eff}}(\mathbf{m}). \tag{31}$$

According to Eq. (31), the LLG equation can be written in the following way:

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times \left(\frac{\delta g}{\delta \mathbf{m}} - \alpha \frac{\partial \mathbf{m}}{\partial t}\right) \text{ in } \Omega. \tag{32}$$

Let us consider an equilibrium configuration $\mathbf{m}_0(\mathbf{r})$ and a small oscillation $\delta\mathbf{m}$ of magnetization with respect to it. The deviation $\delta\mathbf{m}$ can be expressed through the expansion (11), (13), (14), which preserves magnetization magnitude. In this case, the vector field ϑ is also function of time as well as the vector field $\mathbf{v}$.

By substituting this expansion into Eq. (32) and taking the limit for $\varepsilon \to 0$, one ends up with the LLG equation linearized around $\mathbf{m}_0$:

$$\frac{\partial \mathbf{v}}{\partial t} - \alpha \mathbf{m}_0 \times \frac{\partial \mathbf{v}}{\partial t} = \mathbf{m}_0 \times \frac{\delta}{\delta \mathbf{v}} \delta^2 g(\mathbf{m}_0, \mathbf{v}) \quad \text{in } \Omega, \tag{33}$$

where we have used the fact that $\delta g(\mathbf{m}_0, \mathbf{v}) = 0$ (see Eqs. (24) and (25)). In order to compute the derivative of the Hessian functional (19) in the latter equation, one can be easily convinced that:

$$\frac{1}{V_\Omega}\int_\Omega \frac{\delta}{\delta \mathbf{v}} \delta^2 g(\mathbf{m}_0, \mathbf{v}) \cdot \mathbf{v}\, dV = \frac{1}{V_\Omega}\int_\Omega \left(h_0 \mathbf{v} - \mathbf{h}'_{\text{eff}}[\mathbf{v}]\right) \cdot \mathbf{v}\, dV + \frac{1}{V_\Omega}\int_{\partial\Omega} \frac{l_{ex}^2}{2} \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \mathbf{v}\, dS, \tag{34}$$

where

$$\mathbf{h}'_{\text{eff}}[\mathbf{v}] = l_{ex}^2 \nabla^2 \mathbf{v} - \mathcal{N}\mathbf{v} + \kappa_{an} \mathbf{e}_{an} \otimes \mathbf{e}_{an} \cdot \mathbf{v}, \tag{35}$$

(the symbol $\otimes$ denotes the dyadic Kronecker tensor product). Since the homogeneous Neumann boundary condition (29) is assumed to hold for the magnetization dynamics, we restrict our attention to vector fields $\mathbf{v}$ which have zero normal derivative on $\partial\Omega$:

$$\frac{\partial \mathbf{v}}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega. \tag{36}$$

Therefore, it is now possible to express the vector field $\mathbf{h}'_{\text{eff}}$ and the Hessian functional in the following operator form:

$$\mathbf{h}'_{\text{eff}}[\mathbf{v}] = -\mathcal{C}\mathbf{v} \tag{37}$$

$$\delta^2 g(\mathbf{m}_0, \mathbf{v}) = \frac{1}{V_\Omega}\int_\Omega \mathbf{v} \cdot \mathcal{A}_0 \mathbf{v}\, dV \tag{38}$$

where the operators $\mathcal{C}$ and $\mathcal{A}_0$ are properly defined in $\mathbb{H}^1(\Omega)$, which is, roughly speaking, the subspace of $\mathbb{L}^2(\Omega)$ vector fields which admit first partial derivatives (in weak or distributional sense) and these are elements of $\mathbb{L}^2(\Omega)$:

$$\mathcal{C} = -l_{ex}^2 \nabla^2 + \mathcal{N} - \kappa_{an} \mathbf{e}_{an} \otimes \mathbf{e}_{an} \tag{39}$$

$$\mathcal{A}_0 = h_0(\mathbf{r})\mathcal{I} + \mathcal{C}, \tag{40}$$

and $\mathcal{I}$ is the identity operator. By using the Green's identities and the magnetostatic reciprocity theorem[9], it can be easily proved that both the operators $\mathcal{C}$ and $\mathcal{A}_0$, complemented by the boundary conditions (36) are self-adjoint with respect to the classical scalar product:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{V_\Omega}\int_\Omega \mathbf{v}(\mathbf{r}) \cdot \mathbf{w}(\mathbf{r})\, dV, \tag{41}$$

namely:

$$\langle \mathcal{C}\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathcal{C}\mathbf{w} \rangle, \quad \langle \mathcal{A}_0\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathcal{A}_0\mathbf{w} \rangle. \tag{42}$$

We observe that Eq. (38) can be written in compact notation as:

$$\delta^2 g(\mathbf{m}_0, \mathbf{v}) = \langle \mathbf{v}, \mathcal{A}_0 \mathbf{v} \rangle, \tag{43}$$

and that, as long as the condition (27) holds, the operator $\mathcal{A}_0$ is also positive definite.

With the above notations, Eq. (33) can be written as

$$\frac{\partial \mathbf{v}}{\partial t} - \alpha \mathbf{m}_0 \times \frac{\partial \mathbf{v}}{\partial t} = \mathbf{m}_0 \times \left(h_0 \mathbf{v} - \mathbf{h}'_{\text{eff}}[\mathbf{v}]\right) = \mathbf{m}_0 \times \mathcal{A}_0 \mathbf{v}. \tag{44}$$

The latter equations must be complemented with the boundary condition (36).

This result is useful to discuss the stability of the micromagnetic equilibrium $\mathbf{m}_0$. In order to derive some stability conditions, let us rewrite the linearized LLG Eq. (44)

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{m}_0 \times \left(\mathcal{A}_0 \mathbf{v} - \alpha \frac{\partial \mathbf{v}}{\partial t}\right). \tag{45}$$

By scalar multiplying both sides of Eq. (45) by $\left(\mathcal{A}_0 \mathbf{v} - \alpha \frac{\partial \mathbf{v}}{\partial t}\right)$ and integrating over the magnetic body volume Ω, one has:

$$\frac{1}{V_\Omega} \int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \left(\mathcal{A}_0 \mathbf{v} - \alpha \frac{\partial \mathbf{v}}{\partial t}\right) dV = 0. \tag{46}$$

It is easy to see, remembering Eq. (38), that one ends up with the following balance equation:

$$\frac{1}{2} \frac{d}{dt} \delta^2 g(\mathbf{m}_0, \mathbf{v}) = -\frac{1}{V_\Omega} \int_{V_\Omega} \alpha \left|\frac{\partial \mathbf{v}}{\partial t}\right|^2 dV. \tag{47}$$

Now, if the equilibrium configuration $\mathbf{m}_0(\mathbf{r})$ is a minimum of the free energy, then, according to Eq. (27), the Hessian functional $\delta^2 g(\mathbf{m}_0, \mathbf{v})$ is positive definite. Therefore, Eq. (47) states that the rate of change of $\delta^2 g$ must decrease towards zero. This implies that the equilibrium configuration $\mathbf{m}_0(\mathbf{r})$ is also asymptotically stable against small perturbations.

Conversely, as far as the instability of the micromagnetic dynamical system defined by Eq. (45) is concerned, it can be shown that the instability occurs when the Hessian functional $\mathcal{A}_0$ loses its positive definiteness, namely:

$$\exists\, \mathbf{v}' \in T_{\mathcal{M}}(\mathbf{m}_0) : \delta^2 g(\mathbf{m}_0, \mathbf{v}') = \langle \mathbf{v}', \mathcal{A}_0 \mathbf{v}' \rangle < 0. \tag{48}$$

This mathematical property corresponds, for example, to the physical situation in which the applied field is greater than the nucleation field of the magnetic particle [9].

A rigorous proof that Eq. (48) is a sufficient condition for the instability of the system requires the generalization of the Chetaev's theorem [29,30] to the case of infinite-dimensional Banach spaces [31].

4.1. Eigenfunction expansion and orthogonality

In order to determine the normal magnetization oscillations around the stable equilibrium configuration $\mathbf{m}_0$, let us consider, for the time being, only the conservative magnetization dynamics, expressed by Eq. (45) with $\alpha = 0$:

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{m}_0 \times \mathcal{A}_0 \mathbf{v} \text{ in } \Omega, \tag{49}$$

It is clear from Eq. (49) that the right hand side represents a vector field belonging to the tangent space $T_{\mathcal{M}}(\mathbf{m}_0)$, namely, in each point $\mathbf{r} \in \Omega$, it lies in the plane perpendicular to $\mathbf{m}_0(\mathbf{r})$. For this reason, we can rewrite Eq. (49) in the following way:

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{m}_0 \times \mathcal{A}_{0\perp} \mathbf{v} \text{ in } \Omega, \tag{50}$$

where $\mathcal{A}_{0\perp}$ is basically the operator $\mathcal{A}_0$ projected on the tangent space $T_{\mathcal{M}}(\mathbf{m}_0)$:

$$\mathcal{A}_{0\perp} = \mathcal{P}_{\mathbf{m}_0} \mathcal{A}_0, \tag{51}$$

and $\mathcal{P}_{\mathbf{m}_0}$ is the pointwise projection operator on $T_{\mathcal{M}}(\mathbf{m}_0)$, expressed as

$$\mathcal{P}_{\mathbf{m}_0} \mathbf{w} = (\mathcal{I} - \mathbf{m}_0 \otimes \mathbf{m}_0) \cdot \mathbf{w}. \tag{52}$$

As a consequence of the above definitions, the vector fields in the image of the operator $\mathcal{A}_{0\perp}$ are pointwise orthogonal to $\mathbf{m}_0(\mathbf{r})$.

Then, if one considers the vector fields $\mathbf{v}_1, \mathbf{v}_2 \in T_{\mathcal{M}}(\mathbf{m}_0)$, by using the fact that the operator $\mathcal{A}_0$ is self-adjoint, it is immediate to prove that also the projected version $\mathcal{A}_{0\perp}$ is self-adjoint:

$$\langle \mathbf{v}_1, \mathcal{A}_{0\perp} \mathbf{v}_2 \rangle = \langle \mathbf{v}_2, \mathcal{A}_{0\perp} \mathbf{v}_1 \rangle \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in T_{\mathcal{M}}(\mathbf{m}_0). \tag{53}$$

In addition, it happens that

$$\delta^2 g(\mathbf{m}_0, \mathbf{v}) = \langle \mathbf{v}, \mathcal{A}_0 \mathbf{v} \rangle = \langle \mathbf{v}, \mathcal{A}_{0\perp} \mathbf{v} \rangle \quad \forall \mathbf{v} \in T_{\mathcal{M}}(\mathbf{m}_0). \tag{54}$$

Thus, when $\mathbf{m}_0$ is a minimum of the free energy, by using the condition (27) and (43), it is readily seen that the operator $\mathcal{A}_{0\perp}$ is also positive definite:

$$\delta^2 g(\mathbf{m}_0, \mathbf{v}) = \langle \mathbf{v}, \mathcal{A}_0 \mathbf{v} \rangle = \langle \mathbf{v}, \mathcal{A}_{0\perp} \mathbf{v} \rangle > 0 \quad \forall \mathbf{v} \neq 0. \tag{55}$$

We now use the notation previously introduced for the cross product (see Eq. (6)).

$$\mathbf{m}_0(\mathbf{r}) \times \mathbf{w}(\mathbf{r}) = \Lambda(\mathbf{m}_0)\mathbf{w} \tag{56}$$

The operator $\Lambda(\mathbf{m}_0)$ is linear and anti-symmetric and, when is restricted to vector fields in $\mathcal{M}(\mathbf{m}_0)$, is also invertible:

$$\langle \mathbf{v}_1, \Lambda(\mathbf{m}_0)\mathbf{v}_2\rangle = -\langle \mathbf{v}_2, \Lambda(\mathbf{m}_0)\mathbf{v}_1\rangle, \quad \Lambda(\mathbf{m}_0)\Lambda(\mathbf{m}_0) = -\mathcal{I}, \tag{57}$$

which means that the inverse of $\Lambda(\mathbf{m}_0)$ coincides with $-\Lambda(\mathbf{m}_0)$. With this notation the linearized LLG Eq. (50) can be written as

$$\frac{\partial \mathbf{v}}{\partial t} = \Lambda(\mathbf{m}_0)\mathcal{A}_{0\perp}\mathbf{v}. \tag{58}$$

Now we restrict our attention in Eq. (58) to vector fields $\mathbf{v}$ which belong to the tangential manifold $\mathcal{M}(\mathbf{m}_0)$. Thus, by multiplying both sides of Eq. (58) by $\Lambda(\mathbf{m}_0)$ and using the properties (57), one has:

$$\Lambda(\mathbf{m}_0)\frac{\partial \mathbf{v}}{\partial t} = -\mathcal{A}_{0\perp}\mathbf{v}. \tag{59}$$

It is convenient to write Eq. (59) in the frequency domain:

$$-j\omega\Lambda(\mathbf{m}_0)\tilde{\mathbf{v}} = \mathcal{A}_{0\perp}\tilde{\mathbf{v}} \tag{60}$$

where the phasor notation

$$\mathbf{v}(\mathbf{r},t) = \mathrm{Re}\{\tilde{\mathbf{v}}(\mathbf{r})e^{j\omega t}\}, \tag{61}$$

has been used.

The problem of finding the normal oscillations of the magnetization and the natural frequencies for a generic particle consists in determining the values of the frequency ω such that the linearized LLG Eq. (60) has nonzero solution. From the mathematical point of view, Eq. (60) can be regarded as the following (generalized) eigenvalue problem [32]:

$$\mathcal{A}_{0\perp}\tilde{\boldsymbol{\varphi}} = \omega\mathcal{B}_0\tilde{\boldsymbol{\varphi}}, \tag{62}$$

where $\mathcal{B}_0 = -j\Lambda(\mathbf{m}_0)$ (the subindex resembles the dependance on $\mathbf{m}_0$). The latter eigenvalue problem can be equivalently formulated in terms of appropriate sesquilinear forms (see Appendix A.1).

The generalized eigenvalues ω_k and eigenfunctions $\tilde{\boldsymbol{\varphi}}_k$ represent the natural frequencies and the normal oscillation modes of the micromagnetic system, respectively.

One can be easily convinced that, due to the self-adjointness of $\mathcal{A}_{0\perp}$ and the anti-symmetry of Λ, the operators $\mathcal{A}_{0\perp}$ and $\mathcal{B}_0$ are also Hermitian with respect to the standard (complex) inner product of $\mathcal{M}(\mathbf{m}_0)$. One has, for any $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2 \in \mathcal{M}(\mathbf{m}_0)$:

$$\langle \tilde{\mathbf{v}}_1, \mathcal{B}_0\tilde{\mathbf{v}}_2\rangle = \langle \tilde{\mathbf{v}}_2, \mathcal{B}_0\tilde{\mathbf{v}}_1\rangle^*, \tag{63}$$

where now $\langle \tilde{\mathbf{v}}, \tilde{\mathbf{w}}\rangle$ denotes the scalar product (the symbol * indicates complex conjugate):

$$\langle \tilde{\mathbf{v}}, \tilde{\mathbf{w}}\rangle = \frac{1}{V_\Omega}\int_\Omega \tilde{\mathbf{v}}(\mathbf{r})^* \cdot \tilde{\mathbf{w}}(\mathbf{r})\, dV, \quad \tilde{\mathbf{v}}, \tilde{\mathbf{w}} \in \mathcal{M}(\mathbf{m}_0). \tag{64}$$

In addition, it happens:

$$\mathcal{B}_0\mathcal{B}_0 = \mathcal{I}. \tag{65}$$

As a consequence of the above properties, some important facts can be derived about the eigenvalues and the eigenfunctions of the problem (62):

(1) the eigenvalues are all real.
(2) The eigenfunctions related to different and non degenerate eigenvalues are $\mathcal{A}_{0\perp}$-orthogonal.

In order to prove these properties, we refer only to the point spectrum of the eigenvalue problem (62).

In fact, it can be shown (see the Appendix A.2) that the eigenvalue problem has discrete spectrum (the essential spectrum might be only be constituted of one point at infinity) and eigenfunctions which form an orthonormal and complete set in $\mathcal{M}(\mathbf{m}_0) \subset \mathbb{H}^1(\Omega)$. This justifies the computation of the eigenvalues by finite-dimensional approximations of the continuous problem, which will be discussed in the sequel.

Let us consider two eigenfunctions $\tilde{\boldsymbol{\varphi}}_k, \tilde{\boldsymbol{\varphi}}_h$. One can write:

$$\frac{1}{\omega_k}\langle \tilde{\boldsymbol{\varphi}}_h, \mathcal{A}_{0\perp}\tilde{\boldsymbol{\varphi}}_k\rangle = \langle \tilde{\boldsymbol{\varphi}}_h, \mathcal{B}_0\tilde{\boldsymbol{\varphi}}_k\rangle, \tag{66}$$

$$\frac{1}{\omega_h}\langle \tilde{\boldsymbol{\varphi}}_k, \mathcal{A}_{0\perp}\tilde{\boldsymbol{\varphi}}_h\rangle = \langle \tilde{\boldsymbol{\varphi}}_k, \mathcal{B}_0\tilde{\boldsymbol{\varphi}}_h\rangle. \tag{67}$$

By considering the complex conjugate of both sides of Eq. (67), subtracting it from Eq. (66) and taking into account that $\mathcal{A}_{0\perp}$ and $\mathcal{B}_0$ are Hermitian, one ends up with:

$$\left(\frac{\omega_k^* - \omega_h}{\omega_k^*\omega_h}\right)\langle \tilde{\boldsymbol{\varphi}}_h, \mathcal{A}_{0\perp}\tilde{\boldsymbol{\varphi}}_k\rangle = 0. \tag{68}$$

From Eq. (68) one can see that:

- when $k = h$, thanks to the positive definiteness of the operator $\mathcal{A}_{0\perp}$, the imaginary part of the eigenvalue ω_k is necessarily zero:

$$\mathrm{Im}(\omega_k) = \frac{\omega_k - \omega_k^*}{2} = 0. \quad (69)$$

- when $k \neq h$ and the eigenvalues are non degenerate, that is $\omega_k \neq \omega_h$, one ends up with the following orthogonality condition for the eigenfunctions $\tilde{\varphi}_k$ and $\tilde{\varphi}_h$:

$$\langle \tilde{\varphi}_h, \mathcal{A}_{0\perp}\tilde{\varphi}_k \rangle = 0. \quad (70)$$

The latter equation means that $\tilde{\varphi}_k$ and $\tilde{\varphi}_h$ are $\mathcal{A}_{0\perp}$-orthogonal, namely they are orthogonal with respect to the new scalar product defined in $\mathcal{M}(\mathbf{m}_0)$

$$\langle \tilde{\mathbf{v}}, \tilde{\mathbf{w}} \rangle_{\mathcal{A}_{0\perp}} = \langle \tilde{\mathbf{v}}, \mathcal{A}_{0\perp}\tilde{\mathbf{w}} \rangle = \langle \mathcal{A}_{0\perp}\tilde{\mathbf{v}}, \tilde{\mathbf{w}} \rangle, \quad (71)$$

which can be consistently defined since the operator $\mathcal{A}_{0\perp}$ is self-adjoint and positive definite.

In addition, we observe that if ω_k and $\tilde{\varphi}_k$ are respectively the kth eigenvalue and the corresponding eigenfunction of problem (62), then such are also $-\omega_k$ and $\tilde{\varphi}_k^*$:

$$\mathcal{A}_{0\perp}\tilde{\varphi}_k = \omega_k \mathcal{B}_0 \tilde{\varphi}_k \Rightarrow \mathcal{A}_{0\perp}\tilde{\varphi}_k^* = -\omega_k \mathcal{B}_0 \tilde{\varphi}_k^*. \quad (72)$$

In the following, we consider orthonormal eigenfunctions $\tilde{\varphi}_k$ such that:

$$\langle \tilde{\varphi}_h, \mathcal{A}_{0\perp}\tilde{\varphi}_k \rangle = \langle \tilde{\varphi}_h, \tilde{\varphi}_k \rangle_{\mathcal{A}_{0\perp}} = \delta_{hk}, \quad (73)$$

where the symbol δ_{hk} denotes the Kronecker's delta.

The derived orthogonality condition is very important to understand how it is possible to excite selected magnetization modes through the application of appropriate external magnetic fields.

4.2. Perturbation analysis of micromagnetic modes

In the following, we will study the perturbation of the eigenfrequencies ω_k and eigenfunctions $\tilde{\varphi}_k$ which arise from the introduction of a small dissipation in the system, which corresponds to consider Eq. (45) in the frequency domain with $\alpha \neq 0$:

$$-j\omega\Lambda(\mathbf{m}_0) \cdot \tilde{\mathbf{v}} - j\omega\alpha\tilde{\mathbf{v}} = \mathcal{A}_{0\perp}\tilde{\mathbf{v}}. \quad (74)$$

By using the operator notations introduced before, one can write Eq. (74) again in the form of generalized eigenvalue problem:

$$\mathcal{A}_{0\perp}\tilde{\mathbf{v}} = \omega(\mathcal{B}_0 + \delta\mathcal{B})\tilde{\mathbf{v}}, \quad (75)$$

where the operator $\delta\mathcal{B}$ is

$$\delta\mathcal{B} = -j\alpha\mathcal{I}. \quad (76)$$

It can be immediately inferred that the introduction of the damping affects the self-adjointness of the operator $\mathcal{B}_0 + \delta\mathcal{B}$. Therefore, one consequence is that the natural frequencies ω'_k given by the eigenvalues of the problem (75), such that,

$$\mathcal{A}_{0\perp}\tilde{\varphi}'_k = \omega'_k(\mathcal{B}_0 + \delta\mathcal{B})\tilde{\varphi}'_k, \quad (77)$$

will be, in general, complex and will not coincide with the ω_k derived in the conservative case. Moreover, the eigenfunctions of the problem (75) will not be orthogonal anymore.

Nevertheless, for small values of the damping constant α, which is the case in many experimental situations (typically $\alpha \sim 10^{-3}$–10^{-2}), we can make a perturbative study (see Appendix A.3 for a justification) in order to derive the expressions of ω'_k.

We start writing the eigenvalue problem (77) in perturbative form. In fact, we assume that

$$\omega'_k = \omega_k + \delta\omega_k, \quad \tilde{\varphi}'_k = \tilde{\varphi}_k + \delta\tilde{\varphi}_k, \quad \mathcal{A}_{0\perp}\tilde{\varphi}_k = \omega_k \mathcal{B}_0 \tilde{\varphi}_k. \quad (78)$$

With these assumptions, the eigenvalue problem (77) becomes:

$$\mathcal{A}_{0\perp}(\tilde{\varphi}_k + \delta\tilde{\varphi}_k) = (\omega_k + \delta\omega_k)(\mathcal{B}_0 + \delta\mathcal{B})(\tilde{\varphi}_k + \delta\tilde{\varphi}_k). \quad (79)$$

By expanding the latter equation, remembering (78) and retaining only terms up to first-order, one has:

$$\mathcal{A}_{0\perp}\delta\tilde{\varphi}_k = \omega_k \mathcal{B}_0 \delta\tilde{\varphi}_k + \omega_k \delta\mathcal{B}\tilde{\varphi}_k + \delta\omega_k \mathcal{B}_0 \tilde{\varphi}_k. \quad (80)$$

It is convenient to represent the eigenfunctions perturbations $\delta\tilde{\varphi}_k$ by using the unperturbed eigenfunctions $\tilde{\varphi}_k$:

$$\delta\tilde{\varphi}_k = \sum_h \epsilon_{kh}\tilde{\varphi}_h, \tag{81}$$

where the expansion coefficients ϵ_{kh} are the Fourier coefficients given by

$$\epsilon_{kh} = \langle \delta\tilde{\varphi}_k, \tilde{\varphi}_h \rangle_{\mathcal{A}_{0\perp}}. \tag{82}$$

By using the expansion (81) in Eq. (80), one has:

$$\mathcal{A}_{0\perp}\sum_h \epsilon_{kh}\tilde{\varphi}_h = \omega_k\mathcal{B}_0\sum_h \epsilon_{kh}\tilde{\varphi}_h + \omega_k\delta\mathcal{B}\tilde{\varphi}_k + \delta\omega_k\mathcal{B}_0\tilde{\varphi}_k. \tag{83}$$

Now, by scalar multiplying both sides by the eigenfunction $\tilde{\varphi}_k$, remembering the orthonormality condition (73), the expression (76) of the operator $\delta\mathcal{B}$ and solving for the eigenfrequency perturbation $\delta\omega_k$, one ends up with:

$$\delta\omega_k = -\omega_k\frac{\langle\tilde{\varphi}_k, \delta\mathcal{B}\tilde{\varphi}_k\rangle}{\langle\tilde{\varphi}_k, \mathcal{B}_0\tilde{\varphi}_k\rangle} = j\alpha\omega_k^2\frac{\langle\tilde{\varphi}_k, \tilde{\varphi}_k\rangle}{\langle\tilde{\varphi}_k, \mathcal{A}_{0\perp}\tilde{\varphi}_k\rangle} = j\alpha\omega_k^2\langle\tilde{\varphi}_k, \tilde{\varphi}_k\rangle = j\frac{1}{\tau_k}, \tag{84}$$

where $\tau_k = (\alpha\omega_k^2\langle\tilde{\varphi}_k, \tilde{\varphi}_k\rangle)^{-1}$. As expected, we notice that, as a consequence of the stability of $\mathbf{m}_0(\mathbf{r})$, then, according to Eq. (27), the imaginary parts $\delta\omega_k$ of the natural frequencies ω_k' are positive. From the physical point of view, this implies that the magnetization mode $\tilde{\varphi}_k'$ will correspond to a damped oscillation towards the equilibrium configuration with the decay constant τ_k.

Moreover, with some algebraic manipulations, the Fourier coefficients of the eigenfunctions perturbations $\delta\tilde{\varphi}_k$ can be derived:

$$\epsilon_{kh} = j\alpha\frac{\omega_k}{\omega_h - \omega_k}\langle\tilde{\varphi}_k, \tilde{\varphi}_h\rangle \quad k \neq h, \tag{85}$$

$$\epsilon_{kk} = j\frac{\alpha}{2}\langle\tilde{\varphi}_k, \tilde{\varphi}_k\rangle, \quad k = h. \tag{86}$$

We observe that $\delta\omega_k$, as well as ϵ_{kh}, are small quantities if α is small (typically $\alpha \sim 10^{-3}$–10^{-2}) and ω_k is not close to ω_h. Therefore, the coupling between normal modes arising from the introduction of damping is a relatively small effect.

5. Spatially discretized problem

We now proceed to the numerical formulation of the eigenvalue problem (62). To this end, let us suppose to perform a spatial discretization of the magnetic body into N grid points. The values of magnetization in these grid points are typically associated either with the values of magnetization in cells for finite difference methods or with the magnetization values at the nodes for finite element methods.

We denote the magnetization vector associated with the lth grid point by $\mathbf{m}_l(t) \in \mathbb{R}^3$. Analogously, the effective and the applied fields at each grid point will be denoted by the vectors $\mathbf{h}_{\text{eff},l}(t), \mathbf{h}_{a,l}(t)$. In addition to these vectors, we introduce another notation for the mesh vectors which include the information of all grid points of the mesh. In this respect, we will indicate with $\underline{\mathbf{m}}, \underline{\mathbf{h}}_{\text{eff}}, \underline{\mathbf{h}}_{\text{a}}$ the vectors in $\mathbb{R}^{3N}$ given by:

$$\underline{\mathbf{m}} = \begin{pmatrix}\mathbf{m}_1\\ \vdots \\ \mathbf{m}_N\end{pmatrix} \quad \underline{\mathbf{h}}_{\text{eff}} = \begin{pmatrix}\mathbf{h}_{\text{eff},1}\\ \vdots \\ \mathbf{h}_{\text{eff,N}}\end{pmatrix} \quad \underline{\mathbf{h}}_{\text{a}} = \begin{pmatrix}\mathbf{h}_{\text{a},1}\\ \vdots \\ \mathbf{h}_{\text{a,N}}\end{pmatrix}. \tag{87}$$

Usual spatial discretization techniques [28] (e.g. finite elements and finite differences) quite naturally lead to a discretized version of the free energy (2) which has generally the form

$$\underline{g}(\underline{\mathbf{m}}, \underline{\mathbf{h}}_a) = \frac{1}{2}\underline{\mathbf{m}}^T \cdot \underline{C} \cdot \underline{\mathbf{m}} - \underline{\mathbf{h}}_a \cdot \underline{\mathbf{m}}, \tag{88}$$

where $\underline{C}$ is now a $3N \times 3N$ symmetric matrix which describes exchange, anisotropy and magnetostatic interactions. Once the free energy has been discretized, the corresponding spatially discretized effective field $\underline{\mathbf{h}}_{\text{eff}}(\underline{\mathbf{m}}, t)$ can be obtained as

$$\underline{\mathbf{h}}_{\text{eff}}(\underline{\mathbf{m}}, t) = -\frac{\partial \underline{g}}{\partial \underline{\mathbf{m}}} = -\underline{C} \cdot \underline{\mathbf{m}} + \underline{\mathbf{h}}_a(t). \tag{89}$$

We notice that the effective field mathematical structure (31) is formally preserved after the spatial discretization, and the matrix $\underline{C}$ is the discretized version of the self-adjoint integral-differential operator $\mathcal{C}$ defined by (39). By using the above notations, the spatially semi-discretized LLG equation consists in a system of $3N$ coupled ordinary differential equations (ODEs) which, for the generic lth grid point, can be written in the following form:

$$\frac{d}{dt}\mathbf{m}_l = -\mathbf{m}_l \times \left[\mathbf{h}_{\text{eff},l}(\underline{\mathbf{m}}, t) - \alpha\frac{d}{dt}\mathbf{m}_l\right], \tag{90}$$

and for the whole collection of grid points as:

$$\frac{d}{dt}\underline{\mathbf{m}} = -\underline{\Lambda}(\underline{\mathbf{m}}) \cdot \left[\underline{\mathbf{h}}_{\text{eff}}(\underline{\mathbf{m}}, t) - \alpha \frac{d}{dt}\underline{\mathbf{m}}\right], \tag{91}$$

where $\underline{\Lambda}(\underline{\mathbf{m}})$ is the block-diagonal matrix

$$\underline{\Lambda}(\underline{\mathbf{m}}) = \text{diag}(\Lambda(\mathbf{m}_1), \ldots, \Lambda(\mathbf{m}_N)) \tag{92}$$

with blocks $\Lambda(\cdot) \in \mathbb{R}^{3\times3}$ such that $\Lambda(\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \times \mathbf{w}, \forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.

By using the expression of the discrete effective field (89) in the latter equation, one has:

$$\frac{d}{dt}\underline{\mathbf{m}} = \underline{\Lambda}(\underline{\mathbf{m}}) \cdot \left[\frac{\partial \underline{g}}{\partial \underline{\mathbf{m}}} + \alpha \frac{d}{dt}\underline{\mathbf{m}}\right]. \tag{93}$$

By following the same line of reasoning as in the continuum problem, we consider an equilibrium configuration $\underline{\mathbf{m}}_0$ which minimizes the discretized free energy $\underline{g}$ along any admissible small variation $\underline{\mathbf{v}} = (\mathbf{v}_1, \ldots, \mathbf{v}_N)^T$ defined as its counterpart $\mathbf{v}(\mathbf{r})$ in the continuum formulation,

$$\underline{\mathbf{v}} = \underline{\Lambda}(\underline{\vartheta}) \cdot \underline{\mathbf{m}}_0, \tag{94}$$

i.e. such that $\mathbf{v}_l$ is perpendicular to $\mathbf{m}_{0,l}$ according to the rotations described by the collection $\underline{\vartheta} = (\vartheta_1, \ldots, \vartheta_N)^T$.

It can be easily seen that the semi-discretized LLG Eq. (93), linearized around the configuration $\underline{\mathbf{m}}_0$ has the following form:

$$\frac{d}{dt}\underline{\mathbf{v}} - \alpha\underline{\Lambda}(\underline{\mathbf{m}}_0) \cdot \frac{d}{dt}\underline{\mathbf{v}} = \underline{\Lambda}(\underline{\mathbf{m}}_0) \cdot \underline{A}_0 \cdot \underline{\mathbf{v}}, \tag{95}$$

where the matrix $\underline{A}_0$ is the Hessian matrix of $\underline{g}$, evaluated at $\underline{\mathbf{m}}_0$ with respect to the admissible variation $\underline{\mathbf{v}}$:

$$\underline{A}_0 = \underline{H}_0 - \underline{C}, \tag{96}$$

the matrix $\underline{H}_0 = \text{diag}(h_{0,1}\underline{I}_3, \ldots, h_{0,N}\underline{I}_3)$, $h_{0,l}$ is the value of the effective field in the lth grid point due to the equilibrium magnetization $\underline{\mathbf{m}}_0$ and $\underline{I}_3$ is the 3×3 identity matrix. We observe that the matrix $\underline{A}_0$ is defined similarly as its counterpart in the continuum problem (see Eq. (40)).

As before, we consider now only the conservative dynamics $(\alpha = 0)$ and write Eq. (95) in the frequency domain:

$$j\omega\underline{\tilde{\mathbf{v}}} = \underline{\Lambda}(\underline{\mathbf{m}}_0)\underline{A}_0 \cdot \underline{\tilde{\mathbf{v}}}, \tag{97}$$

where the notation $\underline{\tilde{\mathbf{v}}} \in \mathbb{C}^{3N}$ denotes the mesh vector containing all the magnetization phasors $\tilde{\mathbf{v}}_l \in \mathbb{C}^3$ associated with the computational grid points $l = 1, \ldots, N$:

$$\underline{\tilde{\mathbf{v}}} = \begin{pmatrix} \tilde{\mathbf{v}}_1 \\ \tilde{\mathbf{v}}_2 \\ \vdots \\ \tilde{\mathbf{v}}_N \end{pmatrix}. \tag{98}$$

Then, it can be easily inferred that the discretized version of the generalized eigenvalue problem (62) can be written as:

$$\underline{A}_{0\perp} \cdot \underline{\tilde{\mathbf{v}}} = \omega\underline{B}_0 \cdot \underline{\tilde{\mathbf{v}}}. \tag{99}$$

In the latter equation, $\underline{A}_{0\perp} \in \mathbb{R}^{3N\times3N}$ is the discretized version of the operator $\mathcal{A}_{0\perp}$ and, as one can see from Eq. (51), it is given by:

$$\underline{A}_{0\perp} = \underline{P}_{\mathbf{m}_0} \cdot \underline{A}_0, \tag{100}$$

where $\underline{P}_{\mathbf{m}_0} \in \mathbb{R}^{3N\times3N}$ is the discretized projection operator on the vector field $\underline{\mathbf{m}}_0$, given by the following 3×3 block diagonal matrix:

$$\underline{P}_{\mathbf{m}_0} = \text{diag}\left(\underline{I}_3 - \mathbf{m}_{0,1} \cdot \mathbf{m}_{0,1}^T, \ldots, \underline{I}_3 - \mathbf{m}_{0,N} \cdot \mathbf{m}_{0,N}^T\right). \tag{101}$$

The matrix $\underline{B}_0 \in \mathbb{C}^{3N\times3N}$ is the 3×3 block-diagonal matrix obtained in the following way:

$$\underline{B}_0 = -j\underline{\Lambda}(\underline{\mathbf{m}}_0). \tag{102}$$

The generalized eigenvalues ω_k of the problem (99) will give the natural frequencies and the eigenvectors $\underline{\tilde{\varphi}}_k$ will give the normal oscillation modes.

We observe that, as their counterparts $\mathcal{A}_{0\perp}$ and $\mathcal{B}_0$, the matrices $\underline{A}_{0\perp}$ and $\underline{B}_0$ are hermitian and $\underline{A}_{0\perp}$ is also positive definite. Thus, according to the orthogonality property (73) derived for the continuum problem, the eigenfrequencies ω_k of the discretized problem (99) will be real and the eigenvectors will be orthogonal (the notation $\underline{\tilde{\mathbf{v}}}^H$ denotes the complex conjugate matrix transpose):

$$\tilde{\underline{\varphi}}_k^H \cdot \underline{A}_{0\perp} \cdot \tilde{\underline{\varphi}}_h = \delta_{kh}. \tag{103}$$

In addition, when a small damping α is considered, by using the same line of reasoning as in the continuum formulation, one can show that the perturbations $\delta\omega_k$ of the eigenvalues are imaginary and given by the following expressions:

$$\delta\omega_k = j\alpha\omega_k^2 \frac{\tilde{\underline{\varphi}}_k^H \cdot \tilde{\underline{\varphi}}_k}{\tilde{\underline{\varphi}}_k^H \cdot \underline{A}_{0\perp} \cdot \tilde{\underline{\varphi}}_k} = j\alpha\omega_k^2\, \tilde{\underline{\varphi}}_k^H \cdot \tilde{\underline{\varphi}}_k = j\frac{1}{\tau_k}. \tag{104}$$

Moreover, the Fourier coefficients of the eigenfunctions perturbations $\delta\tilde{\underline{\varphi}}_k$ can be derived:

$$\epsilon_{kh} = j\alpha\frac{\omega_k}{\omega_h - \omega_k}\, \tilde{\underline{\varphi}}_k^H \cdot \tilde{\underline{\varphi}}_h \quad k \neq h, \tag{105}$$

$$\epsilon_{kk} = j\frac{\alpha}{2}\, \tilde{\underline{\varphi}}_k^H \cdot \tilde{\underline{\varphi}}_k, \quad k = h. \tag{106}$$

Now we discuss the numerical solution of the generalized eigenvalue problem (99).

First of all, we observe that the dimension of the problem can be reduced to $2N \times 2N$. This allows one to get considerable saving in both storage and computational power. In fact, we know in advance that the normal oscillation modes will be pointwise orthogonal to the equilibrium magnetization configuration. Thus, in an appropriate reference frame pointwise orthogonal to $\mathbf{m}_0$, they will have one component equal to zero.

To this end, let us introduce in the generic lth grid point an orthogonal reference frame whose unit-vectors are given by

$$\mathbf{e}_{1,l} = -\mathbf{m}_{0,l} \times (\mathbf{e}_z \times \mathbf{m}_{0,l}), \quad \mathbf{e}_{2,l} = \mathbf{e}_z \times \mathbf{m}_{0,l}, \quad \mathbf{e}_{3,l} = \mathbf{m}_{0,l}. \tag{107}$$

It is evident, for what we outlined in previous sections, that each vector $\mathbf{v}_l$, represented with respect to this basis, will have the third component equal to zero. The coordinate transformation from the cartesian reference frame and the one defined above is represented for each grid point by the orthonormal 3×3 matrix $\underline{R}_l$ containing the following direction cosines

$$\underline{R}_l = \begin{pmatrix} \mathbf{e}_{1,l}\cdot\mathbf{e}_x & \mathbf{e}_{2,l}\cdot\mathbf{e}_x & \mathbf{e}_{3,l}\cdot\mathbf{e}_x \\ \mathbf{e}_{1,l}\cdot\mathbf{e}_y & \mathbf{e}_{2,l}\cdot\mathbf{e}_y & \mathbf{e}_{3,l}\cdot\mathbf{e}_y \\ \mathbf{e}_{1,l}\cdot\mathbf{e}_z & \mathbf{e}_{2,l}\cdot\mathbf{e}_z & \mathbf{e}_{3,l}\cdot\mathbf{e}_z \end{pmatrix} \tag{108}$$

such that

$$\mathbf{w}_l = \underline{R}_l \cdot \mathbf{v}_l, \tag{109}$$

where $\mathbf{w}_l$ contains the components of the vector associated with the lth grid point with respect to the basis $\{\mathbf{e}_{1,l}, \mathbf{e}_{2,l}, \mathbf{e}_{3,l}\}$. As mentioned before, it happens $\mathbf{w}_l \cdot \mathbf{e}_{3,l} = 0$. The transformation can be expressed also for the mesh vectors by using the 3×3 block-diagonal matrix $\underline{R} = \mathrm{diag}(\underline{R}_1, \ldots, \underline{R}_N)$:

$$\underline{\mathbf{w}} = \underline{R} \cdot \underline{\mathbf{v}}. \tag{110}$$

By multiplying both sides of the problem (99), respectively at left by $\underline{R}^T$ and at right by $\underline{R}$, one obtains:

$$\underline{A}'_{0\perp} \cdot \tilde{\underline{\mathbf{v}}} = \omega\, \underline{B}'_0 \cdot \tilde{\underline{\mathbf{v}}}. \tag{111}$$

where the matrices $\underline{A}'_{0\perp}$ and $\underline{B}'_0$ are:

$$\underline{A}'_{0\perp} = \underline{R}^T \underline{A}_{0\perp} \underline{R}, \tag{112}$$

$$\underline{B}'_0 = \underline{R}^T \underline{B}_0 \underline{R}. \tag{113}$$

By recalling the definition (102), one can be easily convinced that $\underline{B}'_0$ is the $3N \times 3N$ block-diagonal matrix obtained repeating N times the elementary 3×3 blocks

$$\underline{B}_{3\times 3} = \begin{pmatrix} 0 & j & 0 \\ -j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{114}$$

The generalized eigenvalue problem (111) has the same eigenvalues ω_k as (99), but different eigenvectors $\tilde{\underline{\psi}}_k$. Nevertheless, the eigenvectors $\tilde{\underline{\varphi}}_k$ of the original problem are simply given by

$$\tilde{\underline{\varphi}}_k = \underline{R}^T \cdot \tilde{\underline{\psi}}_k. \tag{115}$$

The above coordinate transformation allows one to reduce the size of the problem to $2N \times 2N$ by simply considering the generalized eigenvalue problem for the reduced matrices $\underline{A}''_{0\perp}$ and $\underline{B}''_0$ obtained removing from $\underline{A}'_{0\perp}$ and $\underline{B}'_0$ the rows and columns associated with the third cartesian component of each cell vector. It is worth noting that such discrete eigenvalue problem can be solved by using well-established techniques of numerical linear algebra. In fact, despite the fact that a generalized eigenvalue problem requires, in general, special solution techniques such as, for instance, the QZ method [38], in our case it can be transformed into a standard eigenvalue problem with low computational cost. Since the matrix $\underline{B}_0''$ is hermitian, invertible and coincident with its inverse, one ends up with the standard eigenvalue problem

$$\underline{D}_{0\perp} \cdot \tilde{\underline{\varphi}} = \omega \tilde{\underline{\varphi}}, \quad (116)$$

where the matrix $\underline{D}_{0\perp} = \underline{B}_0'' \underline{A}_{0\perp}''$. We observe that assembling this matrix involves only sparse matrix product operations. The eigenvalue problem (116) is equivalent to the original generalized eigenvalue problem.

6. Numerical results

In order to test the effectiveness and the generality of the method, we have computed the normal oscillation modes and the natural frequencies for two micromagnetic systems with different geometries:

(1) a rectangular nanoscale magnetic thin-film.
(2) an array of 10 magnetic nanospheres assembled in a linear chain.

In the former case the spatial discretization of the eigenvalue problem (62) has been performed by using the finite difference method which is appropriate to deal with structured meshes. In the latter case the spatial discretization of the eigenvalue problem (62) has been performed by using the finite element method which allows one to treat accurately complex geometries.

Both these methods, as already stressed, preserve the structural properties of the continuum problem and, therefore, can be effectively applied to obtain an accurate computation of the normal oscillations.

6.1. Normal oscillations of a Permalloy rectangular magnetic thin-film

We consider a magnetic thin-film 200 nm × 200 nm × 3 nm without magneto-crystalline anisotropy. The non-uniform equilibrium configuration $\mathbf{m}_0(\mathbf{r})$ represents a Landau (vortex) remanent state (see Fig. 1).

This equilibrium configuration has been obtained with dynamical micromagnetic simulations starting from an artificial Landau state. The LLG equation has been numerically integrated with the implicit midpoint rule scheme [23]. The initial

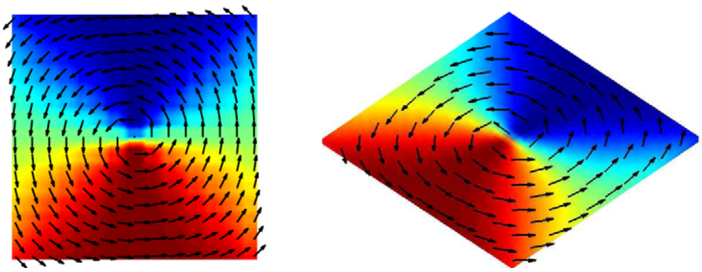

Fig. 1. Equilibrium magnetization configuration computed by micromagnetic simulations. The magnetic thin-film is 200 × 200 × 3 nm^3. The film plane is the x, y plane. The color plot refers to $m_x(\mathbf{r})$. The values of the parameters are: $\alpha = 1, \mu_0 M_s = 1$ T, $\kappa_{an} = 0, l_{ex} = 5.71$ nm. (For interpretation to colours in the figure, the reader is referred to the web version of this paper.)

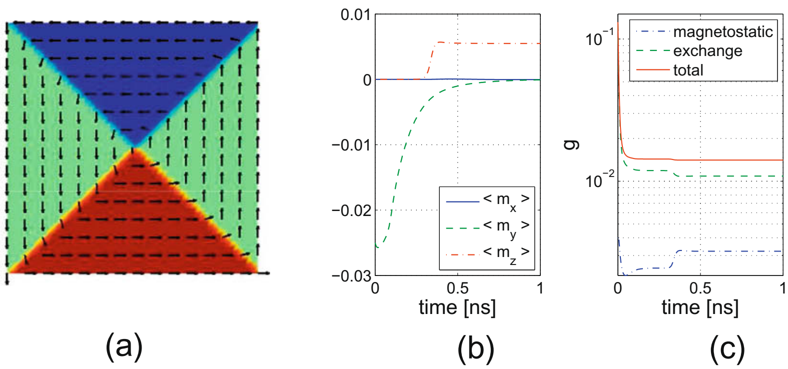

Fig. 2. (a) Initial magnetization state (artificial Landau configuration, the color plot refers to $m_x(\mathbf{r})$). (b) time evolution of the spatially averaged magnetization. (c) time evolution of the discretized free energy. The values of the parameters are the same as in Fig. 1. The total simulated time is 10 ns. (For interpretation to colours in the figure, the reader is referred to the web version of this paper.)

magnetization state and the temporal evolution of the spatially averaged magnetization $\langle \mathbf{m}(t) \rangle$ are reported in Fig. 2. A strong damping constant $\alpha = 1$ has been chosen in order to have fast relaxation toward the equilibrium. It has been checked that after 10 ns the effective torque acting on the magnetization is below machine accuracy. Therefore, we have assumed the magnetization state at the end of the simulation as an equilibrium configuration. The free oscillations of the magnetic system around this remanent state have been computed with the proposed technique.

We have adopted a spatial discretization of the problem (62) on a structured mesh consisted of square prism cells of 5 nm × 5 nm × 3 nm. This mesh size is smaller than the exchange length $l_{ex} = 5.71$ nm in order to guarantee the independence of results on the discretization size [34,35]. The magnetization has been supposed uniform within each cell. The discrete exchange operator has been obtained from a 7-point laplacian difference approximation, whereas the magnetostatic operator has been computed by using a generalization to prism cells of the analytical formulas for cubic cells reported in Ref. [36]. The eigenvalue problem has been solved with an iterative implicitly-restarted Arnoldi method [37] by using a tol-

Table 1
Eigenvalues computations for decreasing discretization sizes of computational cells.

Edge (nm)	f_1 (GHz)	f_2	f_3	f_4	f_5
10	0.04834170116695	4.34719904797588	4.48662612418679	4.53716104452006	4.58012532047085
6.67	0.04776615225695	4.29902353412704	4.48000702275359	4.49926338833020	4.57337813883853
5	0.04772241390201	4.29635175102184	4.47793524071056	4.49560981031406	4.57286540452003
2.5	0.04772202691663	4.29634545523393	4.47791635317348	4.49557475964883	4.57283739100714

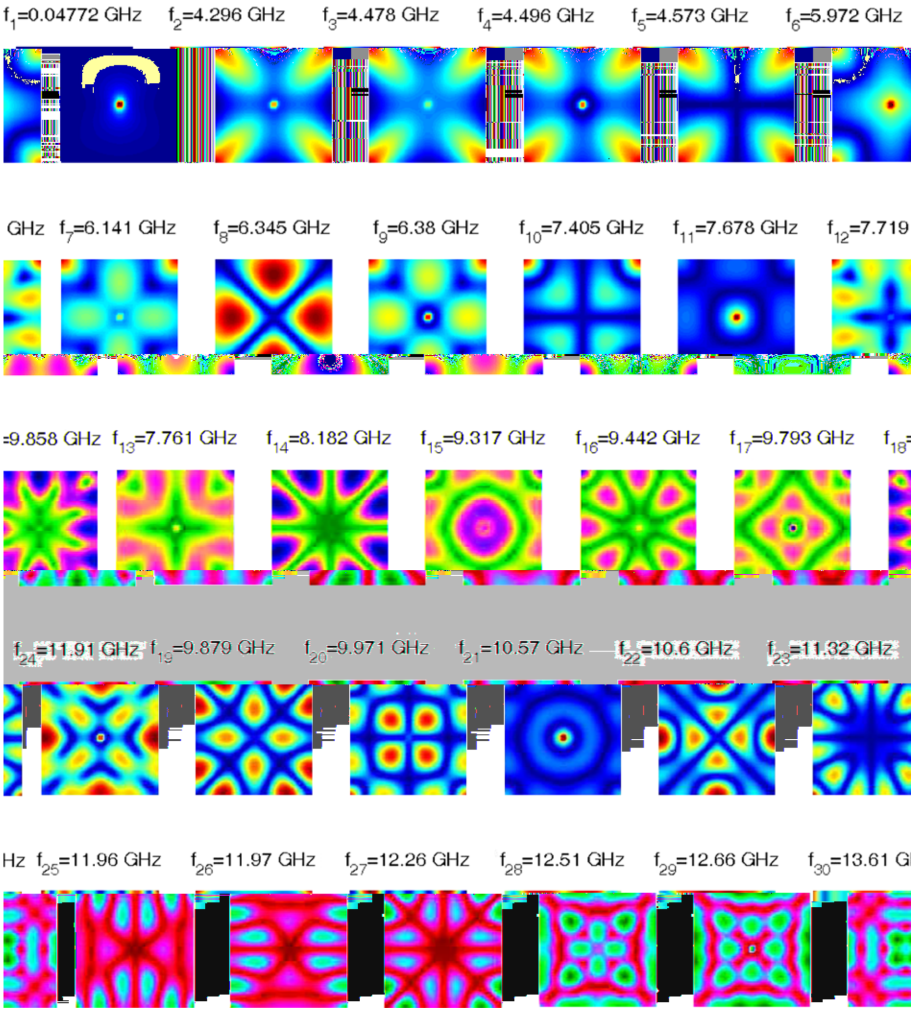

Fig. 3. Numerically computed natural modes and frequencies for a magnetic square thin-film 200 nm × 200 nm × 3 nm without magneto-crystalline anisotropy. The color plot represents the RMS value $\Psi_k(\mathbf{r})$ of the pointwise oscillation amplitude for mode k. The value of the parameters in the computations are $\mu_0 M_s = 1$ T, $l_{ex} = 5.71$ nm, $\kappa_{an} = 0$.

erance within double-precision machine accuracy. This method can be applied to compute generalized eigenvalues and eigenvectors incrementally, in ascending or descending order of magnitude. The proposed formulation of the problem, can be therefore applied to analyze systems with moderately large number of unknowns. The results have been confirmed by direct methods based on the Cholesky factorization [38].

The eigenvalue computations have been repeated for a sequence of decreasing mesh sizes in order to ensure that the results do not depend on the discretization. The results related to the first 5 eigenvalues are reported in Table 1. It is apparent that as the discretization is refined, the computed eigenvalues converge. A discretization size of 5 nm, which is below the exchange length $l_{ex} = 5.71$ nm, is enough to obtain sufficient accuracy.

We have reported in Fig. 3 the frequencies f_k and the computed RMS value $\Psi_k(\mathbf{r})$ of each eigenvector $\tilde{\boldsymbol{\psi}}_k$

$$\Psi_k(\mathbf{r}) = \sqrt{\tilde{\boldsymbol{\psi}}_k \cdot \tilde{\boldsymbol{\psi}}_k^*} \qquad (117)$$

for the first 30 modes.

First, we notice that the first mode has a natural frequency $f_1 \approx 47$ MHz, which is significantly low compared to higher order modes having frequencies in the GHz regime. As one can see from Fig. 3, this mode represents a translational motion of the vortex core. This result is consistent with experimental observations (see, for instance, Ref. [39]).

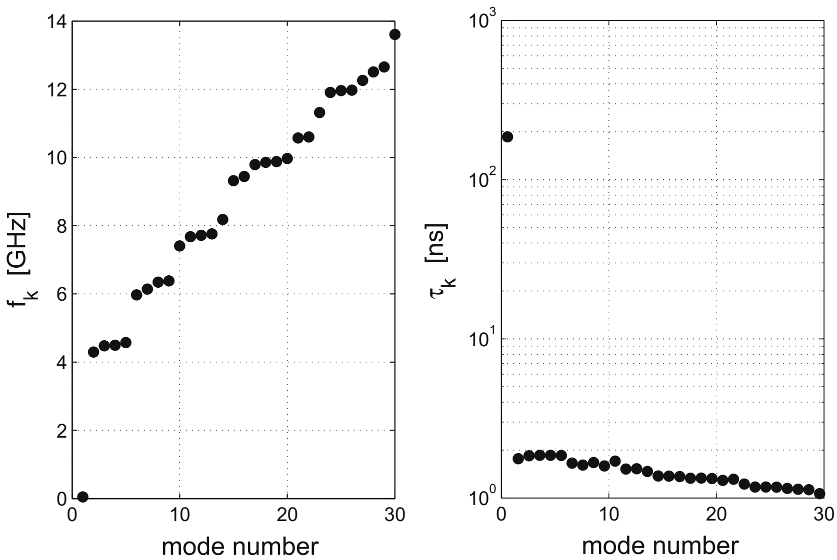

Fig. 4. Numerically computed natural frequencies f_k and time constants τ_k for modes k

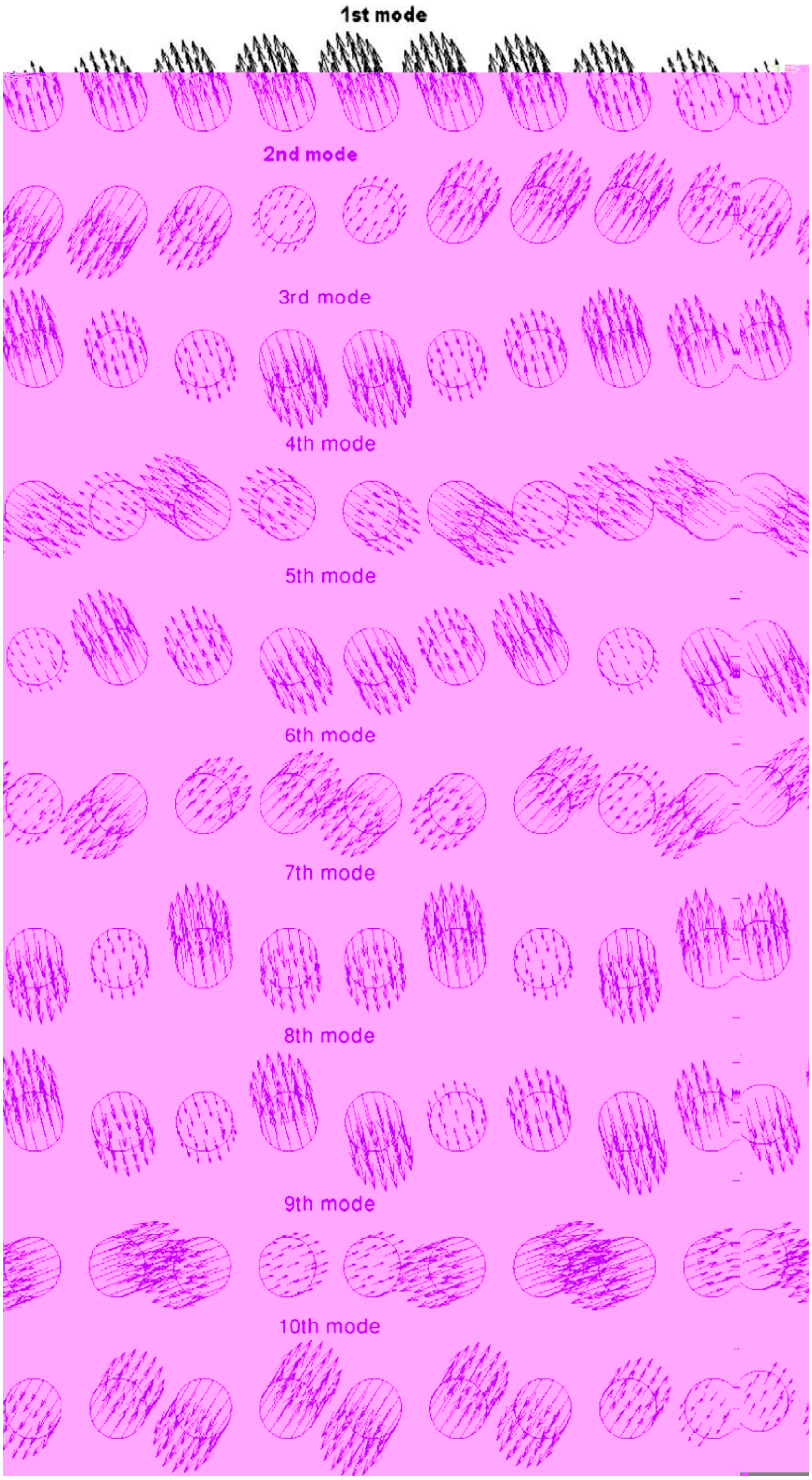

Fig. 6. Numerically computed natural modes $k = 1, \ldots, 10$ for an array of 10 magnetic nanospheres. The values of the parameters are the same as Fig. 5. The arrows refer to magnetization vector field oscillations in the x, y plane at time $t = 0$.

In addition, we observe that the equilibrium configuration $\mathbf{m}_0(\mathbf{r})$ is, at least in principle, symmetric with respect to 90° rotations of the system. Concerning the normal modes, two situations may happen (see Fig. 3):

1. the same symmetry is respected. For example, regarding the modes $k = 1, \dots, 24, 27, \dots, 30$ one can clearly see that $\Psi_k(\mathbf{r})$ are invariant for 90° rotations;
2. the RMS value $\Psi_k(\mathbf{r})$ related to a mode $\tilde{\psi}_k$, which belongs to a pair of degenerate modes $(\tilde{\psi}_k, \tilde{\psi}_h)$, can be obtained from the 90° rotation of $\Psi_h(\mathbf{r})$. For instance, in Fig. 3 one can clearly see that the modes 25 and 26 are degenerate. Then, the RMS value $\Psi_{26}(\mathbf{r})$ can be obtained by rotating $\Psi_{25}(\mathbf{r})$ 90°.

By using Eq. (104), we have computed the time constants τ_k of the first 30 modes. The results, together with the natural frequencies f_k, are reported in Fig. 4. We remark the fact that the decay constant of mode 1 ($\tau_1 \approx 200$ ns) is much larger than other modes', which are in the nanosecond regime.

6.2. Normal oscillations of a linear chain of 10 magnetic nanospheres

We consider an array of 10 spherical magnetic particles disposed in a linear chain along the x axis. The radius of each sphere is $r = 10$ nm and the distance between the centers is $d = 30$ nm.

The equilibrium magnetization configuration is a spatially uniform saturated state along the direction perpendicular to the chain axis. This equilibrium is obtained by applying a DC magnetic field $h_a = 0.35$ perpendicular to the x axis.

Each nanosphere is discretized with a mesh composed of 1279 tetrahedral elements and 308 nodes. The mesh edge is kept below the exchange length $l_{ex} = 7$ nm in order to guarantee the independence of the results on the mesh size. The magnetization is linearly varying within each tetrahedron of the mesh.

The unknowns are the values of the magnetization vector field at the nodes of the mesh. The discrete operator $\underline{A}_{0\perp}$ has been assembled by using finite element micromagnetics[28]. The singularity in the magnetostatic integral operator (22) has been integrated analytically using the results presented in [40].

The numerical results are reported in Figs. 5–7. In Fig. 5 the natural oscillation frequencies of the linear chain are reported. One can clearly see that the first 10 modes correspond to oscillation frequencies around 10 GHz. These low order modes are related to purely dipolar modes in which the magnetization is spatially uniform in each magnetic sphere, as it can be seen in Fig. 6 where the snapshots of the magnetization oscillation in the x, y plane, at time $t = 0$, are reported for these modes.

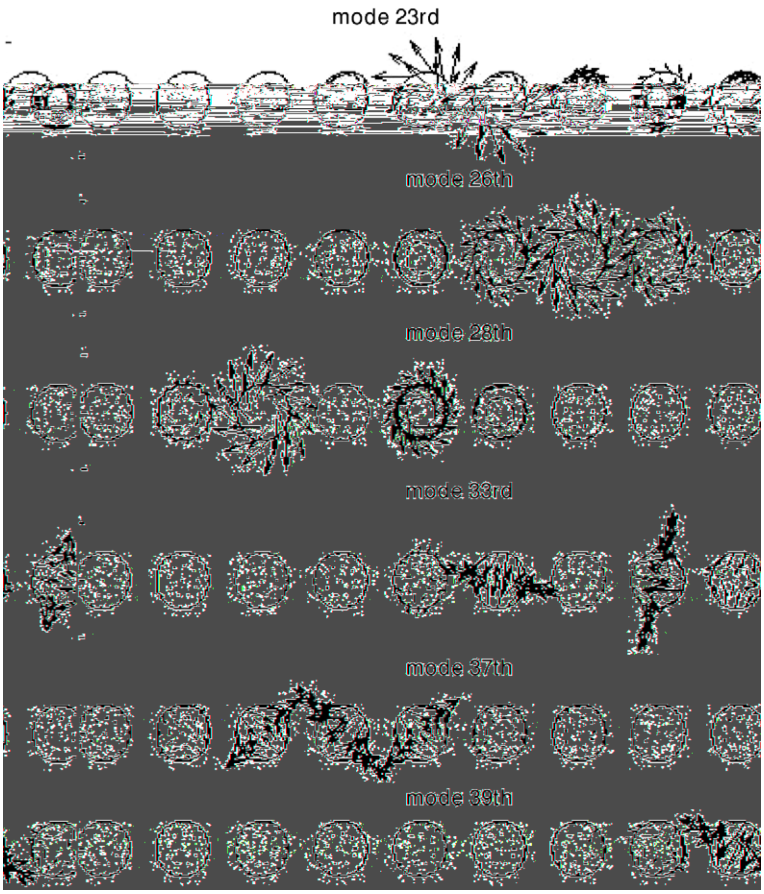

Fig. 7. Numerically computed natural modes for an array of 10 magnetic nanospheres. Magnetization vector field configuration in the x, y plane for higher order modes. Modes 23, 26, 28 are localized vortex modes, modes 33, 37, 39 are localized anti-vortex modes.

We observe that the spectrum of the modes exhibits gaps between modes 10-11 and 20-21. Moreover, the oscillation frequencies of higher order modes are close to 70 GHz. Such high frequencies are associated with spatially non-uniform modes which are strongly localized along the particle chain. For instance, in Fig. 7 one can see that modes 23, 26, 28 are vortex modes localized in different parts of the particle chain. Conversely, modes 33, 37, 39 are localized anti-vortex modes. It is remarkable that these spatially non-uniform modes cannot be analyzed in the framework of dipolar approximation [16].

7. Conclusions

In this paper, a general numerical formulation to analyze the oscillation modes and the natural frequencies of a micromagnetic system directly in the frequency domain has been presented. The method allows one to treat systems with arbitrary geometry and spatially non-uniform equilibrium configurations. Due to the property of magnetization magnitude conservation, which is peculiar of micromagnetics with respect to classical field problems, special attention has been paid in the linearization of the LLG equation. The linearized LLG equation has been recast in the frequency domain as a generalized eigenvalue problem. The main advantages of the formulation are: the frequency domain analysis is the simplest and natural approach to compute the oscillations modes; the generalized eigenvalue problem may be easily discretized by using finite difference or finite element methods depending on the system geometry; the resulting discrete problem can be efficiently solved by using well-established techniques of numerical linear algebra; the discretized model obtained in this way intrinsically preserves the structural properties of the continuum problem. The involved operators are self-adjoint in the lossless limit. The spectral properties of the problem have been studied. In particular, the normal oscillation modes satisfy a special orthogonality condition. This property is very important in order to understand how to excite particular modes of the system. In the limit of small damping, the natural frequencies and modes have been computed by perturbation technique. Furthermore, the introduction of dissipation in magnetization dynamics implies the coupling between the normal oscillation modes. The proposed method can be a powerful tool in the analysis and the design of nanoscale magnetic devices for magnetic recording, spintronics and microwave applications.

Appendix A. Mathematical considerations on the eigenvalue problem

A.1. Formulation of the problem

Let us consider the generalized eigenvalue problem (62):

$$\mathcal{A}_{0\perp}\tilde{\boldsymbol{\varphi}} = \omega\mathcal{B}_0\tilde{\boldsymbol{\varphi}}. \qquad (A.1)$$

The linear operators $\mathcal{A}_{0\perp}$ and $\mathcal{B}_0$ are defined and self-adjoint in $X = \mathcal{M}(\mathbf{m}_0) \subset \mathbb{H}^1(\Omega)$. These operators can be defined by using suitable sesquilinear forms $a(\mathbf{u},\mathbf{v}) : X \times X \to (\mathbf{u},\mathcal{A}_{0\perp}\mathbf{v})$ and $b(\mathbf{u},\mathbf{v}) : X \times X \to (\mathbf{u},\mathcal{B}_0\mathbf{v})$. It can be proved, by using the same line of reasoning as Ref. [41], that $a(\mathbf{u},\mathbf{v})$ and $b(\mathbf{u},\mathbf{v})$ are continuous sesquilinear forms in $X \times X$.

Then, the eigenvalue problem (A.1) can be formulated in terms of the sesquilinear forms $a(\mathbf{u},\mathbf{v})$ and $b(\mathbf{u},\mathbf{v})$:

$$a(\tilde{\boldsymbol{\varphi}},\tilde{\boldsymbol{\psi}}) = \omega b(\tilde{\boldsymbol{\varphi}},\tilde{\boldsymbol{\psi}}) \quad \forall\tilde{\boldsymbol{\varphi}} \in X. \qquad (A.2)$$

A.2. Spectral analysis

The spectral analysis of eigenvalue problems involving self-adjoint (hermitian) operators, in which one of them is definite ($\mathcal{A}_{0\perp}$ in our case), can be completely treated by using the theorems reported in Ref. [32].

In particular, it can be shown that the eigenvalue problem has discrete spectrum (the essential spectrum might be only be constituted of one point at infinity) and eigenfunctions which form an orthonormal and complete set in X. This justifies the computation of the eigenvalues by finite-dimensional approximations of the continuous problem.

In the following, a sketch of the theoretical tools supporting this argument is reported.

It has been shown that, when a micromagnetic equilibrium is considered, the operator $\mathcal{A}_{0\perp}$ is positive definite in X. In addition, it is easy to check that both $a(\mathbf{u},\mathbf{v})$ and $b(\mathbf{u},\mathbf{v})$ are Hermitian in X.

It can be also proved, by using the continuity of $a(\mathbf{u},\mathbf{v})$ and the Friedrichs' inequality ([32] Th. 2.8-2.9 p. 46), that there exists a constant c such that

$$b(\mathbf{u},\mathbf{u}) \leqslant c\, a(\mathbf{u},\mathbf{u}), \qquad (A.3)$$

for all $\mathbf{u}$ in X.

In this situation, it can be shown (see [32, p. 36], Theorem 1.1) that the eigenvalue problem (A.1) is equivalent to the following:

$$\tilde{\boldsymbol{\psi}} = \mu\tilde{\boldsymbol{\psi}}, \qquad (A.4)$$

where the operator $= \mathcal{A}_{0\perp}^{-1}\mathcal{B}_0$ is bounded on X.

Moreover, under the hypothesis that the boundary of the domain Ω is sufficiently regular, Theorems 2.9 and 2.5 in [32] prove that the eigenvalues of the operator exist and form a non increasing sequence $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n \geqslant \cdots$.

Analogously, it can be proved that the eigenvalues of the operator $-$ exist and form a nondecreasing sequence $-\mu_1^- \leqslant -\mu_2^- \leqslant \cdots \leqslant -\mu_n^- \leqslant \cdots$. Both of these sequences converge to zero. Then, by using Theorem 2.6 in [32] it can be proved that any point of the essential spectrum of $-$ cannot lie above (below) any of the eigenvalues $\mu_j = 1/\omega_j$ $(-\mu_j^- = -1/\omega_j)$, ω_j being the natural frequencies.

Therefore, in our case the essential spectrum may be constituted by the only point $\mu = 0$, that corresponds to a frequency $\omega \to \infty$.

Finally, still by using Theorem 2.9, it is possible to show that $b(\mathbf{u},\mathbf{u})$ is completely continuous with respect to $a(\mathbf{u},\mathbf{u})$. This is enough to apply Theorem 3.1 of [32] to say that any element $\mathbf{v}$ in X can be decomposed in Fourier series of the eigenfunctions of , which form an orthonormal and complete set in X. In addition, a consequence of Theorem 3.1 is that the operator is compact.

This results allow to compute the eigenvalues of the problem (A.1) by finite-dimensional approximations of the operators $\mathcal{A}_{0\perp}$ and $\mathcal{B}_0$.

A.3. Perturbation analysis

Let us consider a perturbation of problem (A.1) in the form (77), which depends on the small parameter $\alpha \ll 1$:

$$\mathcal{A}_{0\perp}\tilde{\boldsymbol{\psi}}_\alpha = \omega(\alpha)\mathcal{B}(\alpha)\tilde{\boldsymbol{\psi}}_\alpha, \quad (A.5)$$

where $\mathcal{B}_0(\alpha)$ is

$$\mathcal{B}(\alpha) = \mathcal{B}_0 - j\alpha\mathcal{I}, \quad (A.6)$$

where $\mathcal{I}$ is the identity operator in X. The sub-index α also indicates the dependence of the eigenfunction on α. It is apparent that the perturbed operator $\mathcal{B}(\alpha)$ is bounded-holomorphic [33], since it can be expressed as a convergent power series of bounded operators (in our case it is composed only of two terms, both of them bounded). It can be also proved that $\mathcal{B}(\alpha)$ is invertible with inverse $\mathcal{B}^{-1}(\alpha)$.

These hypotheses are enough (p. 419 of [33]) to ensure that the problem (A.5) is equivalent to the following:

$$\mathcal{A}_b(\alpha)\tilde{\boldsymbol{\psi}}_\alpha = \mathcal{B}^{-1}(\alpha)\mathcal{A}_{0\perp}\tilde{\boldsymbol{\psi}}_\alpha = \omega(\alpha)\tilde{\boldsymbol{\psi}}_\alpha, \quad (A.7)$$

where $\mathcal{A}_b(\alpha)$ is a closed operator forming a holomorphic family. Thus, the analytic perturbation theory (Chapter VII of Ref. [33]) is applicable to $\mathcal{A}_b(\alpha)$. In particular, one result is that any isolated eigenvalue of the unperturbed operator $\mathcal{A}_b(\alpha = 0)$ can be continued as an analytic function $\omega(\alpha)$, which is an eigenvalue of $\mathcal{A}_b(\alpha)$ for each α. Similar results hold for the eigenfunctions $\tilde{\boldsymbol{\psi}}_\alpha$.

The above arguments imply that perturbation theory can be applied to the eigenvalue problem (A.1).

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